

10.4

Ex 2.2 - Lake pollution model:

The problem may be considered on the compartmental model with the lake as the compartment.

To describe such a problem, let us consider the example of problem of salt dissolved in a tank.

Ex 2.2.1, Find a differential equation for the amount of salt in the tank at any time t . The concentration can be defined as the mass per unit volume of mixture.

Soln Let us use the compartmental model with the tank as the compartment.

Let $x(t)$ be the mass of salt in the tank at time t . Let us draw a diagram of the problem.

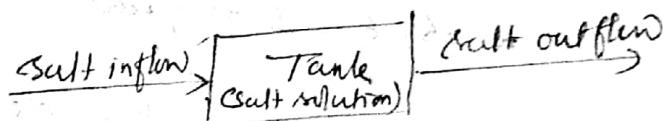


Fig 10.4 Then the balance law to the mass of salt is given by described as follows:

$$\left\{ \begin{array}{l} \text{rate of change} \\ \text{of mass of salt} \\ \text{In tank} \end{array} \right\} = \left\{ \begin{array}{l} \text{rate} \\ \text{mass of} \\ \text{salt enters} \\ \text{tank} \end{array} \right\} - \left\{ \begin{array}{l} \text{rate} \\ \text{mass of} \\ \text{salt leaves} \\ \text{tank} \end{array} \right\}$$

→ { A large tank contains 100 litres of salt water (also called brine). Initially no kg of salt is dissolved. Salt water flows into the tank at the rate of 10 litres per minute and the concentration $C_{in}(t)$ (kg of salt per litre) of the incoming water salt mixture varies with time. It is assumed that the solution in the tank is thoroughly mixed and that the salt solution flows out, at the same rate at which it flows in, i.e. The volume of salt water mixture in the tank remains constant. }

Now, salt is added salt-water enters the tank at the rate 10 litres/minute.
 \therefore Salt is added to the mixture at the rate $10 \cdot C_{in}(t)$ per minute.

i. rate salt enters tank } = $10 C_{in}(t)$.

The rate at which salt is leaving the tank = flow rate \times concentration of salt in the tank

$$= 10 \times \frac{x(t)}{100}$$

$$\therefore \text{rate of salt leaving tank} = \frac{x(t)}{10} \text{ kg/minute}$$

∴ mathematically, (1) can be expressed as

$$\frac{dx}{dt} = 10 C_{in}(t) - \frac{x(t)}{10} \quad \text{--- (2)}$$

To solve equation (2), an initial

condition is required. Let x_0 be the amount of salt at time $t=0$ (in kg).

The IVP (Initial value Problem)

becomes

$$\frac{dx}{dt} = 10C_{in}(t) - \frac{x(t)}{10}, \quad x(0) = x_0 \quad (3)$$

$$\text{or } \frac{dx}{dt} + \frac{1}{10}x(t) = 10C_{in}(t)$$

It is a 1st order linear differential

equation.

$$-I.P.F = e^{\int \frac{1}{10} dt} = e^{\frac{t}{10}}$$

$$\therefore \text{solution} \quad x = \int_{x_0}^t 10C_{in}(t') e^{\frac{t-t'}{10}} dt'$$

$$\text{or } x e^{-\frac{t}{10}} = 10 \int_0^t C_{in}(t') e^{\frac{t-t'}{10}} dt'$$

$$\Rightarrow x(t) = 10e^{-\frac{t}{10}} \int_0^t C_{in}(t') e^{\frac{t-t'}{10}} dt' + x_0 e^{-\frac{t}{10}} \quad (4)$$

Suppose the concentration of the incoming salt solution = constant, C_{in} .

$$\text{i.e. } C_{in}(t) = C_1$$

$$\text{then (4) becomes } x(t) = x_0 e^{-\frac{t}{10}} + 10e^{-\frac{t}{10}} \int_0^t C_1 e^{\frac{t-t'}{10}} dt'$$

$$= x_0 e^{-\frac{t}{10}} + 10e^{-\frac{t}{10}} C_1 \left[10e^{\frac{t}{10}} \right]_0^t$$

$$\Rightarrow x(t) = x_0 e^{-\frac{t}{10}} + 100 e^{-\frac{t}{10}} [e^{\frac{t}{10}} - 1] \\ = x_0 e^{-\frac{t}{10}} + 10e^{-\frac{t}{10}} (1 - e^{-\frac{t}{10}}) \quad \textcircled{5}$$

Suppose $C_{in}(t)$ is a sinusoidal function, say

$$C_{in}(t) = 0.2 + 0.1 \sin t$$

Then (4) gives

$$x(t) = 10 e^{-\frac{t}{10}} \int_0^t (0.2 + 0.1 \sin t) e^{\frac{t}{10}} dt + x_0 e^{-\frac{t}{10}}$$

$$= x_0 e^{-\frac{t}{10}} + 10 e^{-\frac{t}{10}} \left[\frac{1}{10} \cdot 6.2 \cdot \frac{e^{\frac{t}{10}}}{\frac{t}{10}} \right]_0^t$$

$$+ 10 e^{-\frac{t}{10}} \cdot (0.1) \cdot \frac{1}{10} \left[\frac{e^{\frac{t}{10}}}{\frac{t}{10}} \sin t - \frac{e^{\frac{t}{10}}}{\frac{t}{10}} \cos t \right]_0^t$$

$$= x_0 e^{-\frac{t}{10}} + 10 e^{-\frac{t}{10}} \left[\frac{t}{20} e^{\frac{t}{10}} - 20 \right]$$

$$+ \frac{10}{101} \left[\sin t - 10 \cos t \right] + 10 e^{-\frac{t}{10}}$$

$$= x_0 e^{-\frac{t}{10}} + 20 - 20 e^{-\frac{t}{10}}$$

$$+ \frac{10}{101} (\sin t - 10 \cos t)$$

$$+ \frac{100}{101} e^{-\frac{t}{10}}$$

$$= x_0 e^{-\frac{t}{10}} + 20 + \frac{10}{101} \left[\sin t - 10 \cos t \right] - 192 e^{-\frac{t}{10}}$$

$$\text{Let } I = \int e^{\frac{t}{10}} dt$$

$$= 10 e^{\frac{t}{10}} \sin t$$

$$= \int e^{\frac{t}{10}} \cdot 10 e^{\frac{t}{10}} dt$$

$$= 10 e^{\frac{t}{10}} \sin t$$

$$- 10 \cdot \left[10 e^{\frac{t}{10}} \cos t \right] - (\sin t) \cdot 10 e^{\frac{t}{10}} dt$$

$$= 10 e^{\frac{t}{10}} \sin t - 100 e^{\frac{t}{10}} \cos t$$

$$\rightarrow 100 \int e^{\frac{t}{10}} \sin t dt$$

$$\Rightarrow I = \frac{10}{101} \left[e^{\frac{t}{10}} \sin t - 10 e^{\frac{t}{10}} \cos t \right]$$

$$= \frac{100}{101} e^{\frac{t}{10}} - 20 e^{\frac{t}{10}}$$

$$= \frac{1920}{101} e^{\frac{t}{10}} - \frac{2020}{1920} e^{-\frac{t}{10}}$$

$$\text{--- \textcircled{6}}$$

As $t \rightarrow \infty$, The effect of the initial condition on the solution decreases and becomes negligible in equations (4), (5) and (6). The solution has two parts: one provides response to the initial state and the other response to the input.

2.2.1 A lake pollution model

We now return to our original problem of pollution in a lake.

We shall apply the above concept or theory for investigation of the changing concentration of a pollutant.

At first we shall make the following assumptions:

- * The volume of the lake is constant,

- * say V .

- * The pollution is uniform throughout.

Let $m(t)$ be the concentration of the pollutant at any time t . Let f be the rate of flow of water entering the lake in m^3/day .

Since, volume of lake is constant,

$\left\{ \begin{array}{l} \text{flow of mixture into lake} \\ \text{flow of mixture out of lake} \end{array} \right\} = \left\{ \begin{array}{l} \text{flow of mixture out of lake} \end{array} \right\}$

Applying balance law, to the mass of pollutant $m(t)$, we get the word-equation of the process as

$\left\{ \begin{array}{l} \text{rate of change of mass of pollutant in lake} \end{array} \right\} = \left\{ \begin{array}{l} \text{rate at which the pollutant enters the lake} \end{array} \right\} - \left\{ \begin{array}{l} \text{rate at which the pollutant leaves the lake} \end{array} \right\}$

The mathematical expression becomes

$$\frac{dm}{dt} = f c_{in} - f \frac{m(t)}{V} \quad \rightarrow \textcircled{1}$$

where c_{in} is the concentration (in unit mass per unit volume, such as g/m³) of the pollutant in the flow entering the tank.

$$\Rightarrow m(t) = x(t)V \quad \{ \because V = \text{constant} \}$$

$$\therefore \frac{dm}{dt} = \frac{dx}{dt} \cdot V$$

Let us assume, for simplicity, that V is constant.

$$\therefore \textcircled{1} \text{ becomes, } \frac{dx}{dt} = f c_{in} - f \frac{m(t)}{V}$$

$$V \frac{dx}{dt} = f c_{in} - f \frac{m(t)}{V}$$

$$\Rightarrow \frac{dx}{dt} = \frac{f c_{in} - f \frac{m(t)}{V}}{V}$$

$$= \frac{f}{V} c_{in} - f \frac{x(t)}{V}$$

$$\Rightarrow \frac{dx}{dt} = \frac{f}{V} [c_{in} - x(t)] \quad \rightarrow \textcircled{2}$$

If f , the rate of flow is constant, then

$$\int \frac{dx}{c_{in} - x} = f \frac{t}{V} dt$$

$$\text{Integrating, } -\ln(c_{in} - x) = \frac{f}{V} t + C_1, \quad C_1 \text{ is constant}$$

$$= \frac{f}{V} t + \ln C, \text{ say}$$

$$\Rightarrow \ln(c_{in} - x) - \ln C = -\frac{f}{V} t$$

$$\Rightarrow \ln \frac{c_{in} - x}{C} = -\frac{f}{V} t$$

$$\Rightarrow \frac{c_{in} - x}{C} = e^{-\frac{f}{V} t}$$

$$\Rightarrow x(t) = c_{in} - C e^{-\frac{f}{V} t}$$

$$\Rightarrow x(t) = C_n - e^{-k} e^{-\frac{t}{V}} \text{ where } e^{-k} = c.$$

If initially, $x(0) = x_0$ i.e. $x(0) = x_0$,

i.e. when $t=0$, $x=x_0$, then

$$x(0) = C_n - e^{-k} e^0$$

$$\Rightarrow x_0 = C_n - e^{-k}$$

$$\therefore x(t) = C_n - (C_n - x_0)e^{-\frac{t}{V}} \quad \text{--- (3)}$$

Thus, the solution has two parts:

The contribution from initial data, namely $x_0 e^{-\frac{t}{V}}$ and the other is the contribution from the pollution inflow to the system, namely $(C_n - C_n e^{-\frac{t}{V}})$.

Also, we observe that ~~as $t \rightarrow \infty$~~ ,

as $t \rightarrow \infty$, $x(t) \rightarrow C_n$

$$\therefore \lim_{t \rightarrow \infty} [C_n - (C_n - x_0)e^{-\frac{t}{V}}] = C_n. \quad \text{--- (4)}$$

\therefore The concentration on the lake increases or decreases ~~at~~ steadily to the limit -

Ques. How long will it take for the lake's pollution level to reach 5% of its initial level if only fresh water flows into the lake?

Soln. putting $C_n = 0$ in (3), we get

$$x(t) = x_0 e^{-\frac{t}{V}}. \quad \text{--- (5)}$$

$$(5) \text{ ii) } \frac{x(t)}{x_0} = e^{-\frac{f}{V} t}$$

$$\Rightarrow -\frac{f}{V} t = \log \frac{x(t)}{x_0}$$

$$\Rightarrow t = -\frac{V}{f} \log \frac{x(t)}{x_0}$$

$$x = 5\% x_0$$

$$\therefore x = 0.05 x_0$$

$$\therefore t = -\frac{V}{f} \log \frac{0.05 x_0}{x_0}$$

$$= -\frac{V}{f} \log 0.05$$

$$= \frac{V}{f} \log \frac{1}{0.05}$$

$$= \frac{V}{f} \log 20$$

$$= \frac{3V}{f} \cdot 11$$

~~10.81~~ Case Study: Lake Burley Griffin:

Lake Burley Griffin is an artificial lake in the capital city Canberra, of Australia. It was created in 1962 for both recreational and aesthetic purposes. In 1974, it was found that the lake water became polluted and it was not safe for recreational use. It happened due to the discharge of untreated sewage into the feeder rivers of the lake.

In 1974, the mean concentration of the bacteria faecal coliform count was found to be 10^7 per m^3 . The limit of bacteria for suitable recreational

condition should be not more than 4×10^6 bacteria per m^3 . [The sample amount is 10% of total sample over a period of 30 days.]

The problem is to find how the pollution level can be brought under control i.e. the level should be below the safety threshold.

This system can be modelled under the following assumptions as the equation (2) above.

- * Flow onto the lake is equal to flow out of the lake.

- * The volume V of the lake is constant and equal to $28 \times 10^6 m^3$ approximately.

- * The pollution concentration is uniform throughout the lake.

Then equation of the pollution is described by

$$\frac{dx}{dt} = \frac{f}{V} (C_{in} - x), \quad \text{--- (2)}$$

The solution is

$$x(t) = C_{in} - (C_{in} - x_0)e^{-\frac{ft}{V}} \quad \text{--- (2)}$$

The pollution can be dropped below the safety level of threshold with by entering fresh water to the lake ($C_{in}=0$) with a mean monthly

flow of $4 \times 10^6 \text{ m}^3$ per month and initial faecal coliform count of 10^7 bacteria per m^3 (as was measured in 1974). Then the lake pollution level of the lake can be dropped to below the safety level after 6 months.

From the solution (2), it follows that the same as time increases, the concentration of the pollutant in the lake will approach the concentration of the polluted water that enters the lake. Also, the pollution level doesn't depend on the initial level of pollution. ~~If $C_0 > C_{in}$~~

The pollution level decreases monotonically if $\kappa_0 > \kappa_{in}$ and increases if $\kappa_0 < \kappa_{in}$ until it reaches C_{in} :